SHARP MEASURE ZERO

- The notion
- Hausdorff measures
- Borel-like definition
- Products
- Galvin-like game
- Partition arrow

SHARP MEASURE ZERO

DEF A set $S \subseteq X$ in a complete metric space X has **SHARP MEASURE ZERO** if, for each gauge g there is a σ-compact set K, $S \subseteq K$, $\mathcal{H}^g(K) = 0$.

UPPER HAUSDORFF MEASURE

$\longrightarrow$ $\overline{\mathcal{H}}^g_\delta(E) = \inf\left\{ \sum_n g(\operatorname{diam} E_n) : \{E_n\} \text{ is a } \underline{\text{finite}} \text{ cover of } E \right\}$

$\longrightarrow$ $\overline{\mathcal{H}}^g_0(E) = \sup_{\delta > 0} \mathcal{H}^g_\delta(E)$

$\hookrightarrow$ finitely subadditive but not a measure

$\longrightarrow$ $\overline{\mathcal{H}}^g(E) = \inf\left\{ \sum \overline{\mathcal{H}}^g_0(E_n) : \{E_n\} \text{ a countable cover of } E \right\}$

$\hookrightarrow$ measure

$\longrightarrow$ X complete : $\overline{\mathcal{H}}^g(E) = \inf\left\{ \mathcal{H}^g(K) : K \supseteq E \ \sigma\text{-compact} \right\}$

$\longrightarrow$ $\overline{\mathcal{H}}^g(E) < \infty \Rightarrow E$ σ-totally bounded

SHARP MEASURE ZERO — AGAIN

DEF X has **SHARP MEASURE ZERO** ($Smz^{\#}$) if $\overline{\mathcal{H}}^{g}(X) = 0$ for every gauge g.

→ $Smz^{\#}(X)$ – all $Smz^{\#}$ subsets of X

→ σ-ideal

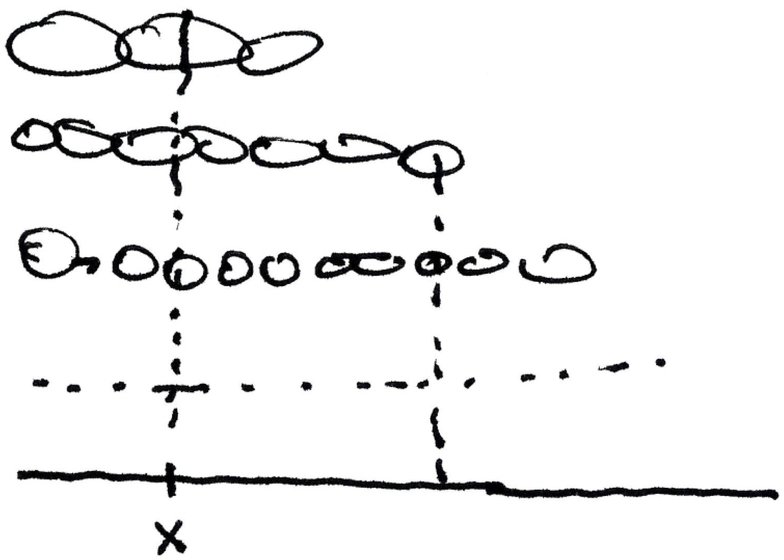
→ preserved by uniformly continuous maps

→ but if X is σ-compact or complete, then $S \subseteq X$ $Smz^{\#} \Rightarrow f(S)$ $Smz^{\#}$ for each continuous map f on X.

→ X $Smz^{\#} \Rightarrow$ X σ-totally bounded

BOREL-LIKE DEFINITION

RECALL : $\mathcal{H}^g(X)=0 \dots \exists\ \lambda$-cover $\sum_n g(\operatorname{diam} E_n)<\infty$



Each x is covered by almost all levels.

→ $\{G_n\}$ is a γ-cover of X if $\forall x\ \forall_n^\infty\ x\in G_n$

→ $\{E_n\}$ is a γ-groupable cover if there is a partition $\omega = I_0 \cup I_1 \cup \dots$ into finite sets s.t. $\{\bigcup_{n\in I_k} E_n : k\in\omega\}$ is a γ-cover.

LEMMA $E\in \mathcal{N}_\sigma(\overline{\mathcal{H}}_0^g) \iff E$ has a γ-groupable cover s.t. $\sum_n g(\operatorname{diam} E_n)<\infty$

(almost $\overline{\mathcal{H}}^g(E)=0$)

BOREL-LIKE DEFINITION

$\mathcal{N}_\sigma(\overline{\mathcal{H}}_0^g)$ .... σ-ideal generated by the sets with $\overline{\mathcal{H}}_0^g(E) = 0$.

LEMMA $E \in \mathcal{N}_\sigma(\overline{\mathcal{H}}_0^g) \iff$

$\iff \exists$ γ-groupable cover $\{E_n\}$ s.t. $\sum_n g(dE_n) < \infty$

THM (Oz 2019) TFAE

(i) X is $Sm_Z^{\#}$

(ii) for each $\langle \varepsilon_n \rangle$ there is an $\langle \varepsilon_n \rangle$-fine γ-groupable cover of X

PRODUCTS

QUESTION (Schepers 1999) When is a product of Smz sets Smz? (CH example: $X \times X$ not Smz)

THM (Oz 2019) TFAE

(i) X is Smz$^{\#}$

(ii) $\check{H}^q(X \times Y) = 0$ whenever $\check{H}^q(Y) = 0$

(iii) $\check{H}^1(X \times E) = 0$ for all $E \in \mathcal{E}$

THM (Oz 2019)

$\longrightarrow$ X Smz$^{\#}$, Y Smz $\Rightarrow$ $X \times Y$ Smz

$\longrightarrow$ X Smz$^{\#}$, Y Smz$^{\#}$ $\Rightarrow$ $X \times Y$ Smz$^{\#}$

PRODUCTS

- $(\operatorname{cov} \mathcal{M} = \mathfrak{c})$ $\exists X \subseteq \mathbb{R}$ X Smz

  $X \times X$ not Smz
  $\Rightarrow X$ not Smz$^{\#}$

- $X, Y \subseteq G$ Smz$^{\#}$ $\Rightarrow$ $X + Y$ Smz$^{\#}$

- $X \subseteq G$ Smz$^{\#}$ $\Rightarrow \exists H \subseteq G$ subgroup
  $X \subseteq H$, H Smz$^{\#}$

- TFAE for $X \subseteq \mathbb{R}^2$
  - X is Smz$^{\#}$
  - all projections on lines are Smz$^{\#}$
  - two projections on lines are Smz$^{\#}$

CORAZZA'S MODEL

Corazza 1989 : forcing extension

- let R be the ground model reals
- $|R| = \omega_1$
- R is not meager in the extension
- $Sm_2(\mathbb{R}) = [\mathbb{R}]^{\leq \omega_1}$

PROP

- R is not $Sm_2^{\#}$

  $\vdash$ not meager $\dashv$

- $R \times X$ is Sm_2 for all $X \subseteq \mathbb{R}$ Sm_2

  $\vdash Y\ Sm_2 \Rightarrow |Y| \leq \omega_1 \Rightarrow |R \times X| \leq \omega_1 \Rightarrow$

  $R \times X\ Sm_2$ $\dashv$

QUESTION: Characterize sets S such that

$$X\ Sm_2 \Rightarrow S \times X\ Sm_2$$

GALVIN-LIKE GAME

$G^{\#}$

I	ε_0	ε_1	ε_2	.....
II	E_0	E_1	E_2	

Rules : $\longrightarrow$ $\varepsilon_n > 0$

$\longrightarrow$ $\operatorname{diam} E_n \leq \varepsilon_n$

II wins if $\{E_n\}$ is a γ-groupable cover

THM (Oz 2019) Let X be a separable metric space.

TFAE: (i) X is $\mathrm{Sm}_2^{\#}$

(ii) I does not have a winning strategy.

Note : X is not supposed σ-totally bounded.

PARTITION ARROW

$\longrightarrow$ Ω^{unif} ... uniform ω-covers

$\longrightarrow$ $\mathcal{O}^{\gamma\text{-gp}}$ ... γ-groupable covers

THM (Oz 2022+) TFAE

(i) X is $Sm_{\mathbb{Z}}^{\#}$

(ii) $\Omega^{unif} \longrightarrow (\mathcal{O}^{\gamma\text{-gp}})^2_2$ and X is σ-t.b.

(iii) $\Omega^{unif} \longrightarrow (\mathcal{O}^{\gamma\text{-gp}})^2_k$ $\forall k$ and X is σ-t.b.

MEAGER-ADDITIVE SETS

DEF $X \subseteq G$ is meager-additive if XM is meager for each $M \subseteq G$ meager

→ $\mathcal{M}^*(G)$ ... all meager-additive sets in G

→ σ-ideal

→ GMS Theorem for $\mathrm{Sm}\mathbb{Z}^{\#}$ – 2^ω

→ $\mathcal{E}$-additive sets

→ GMS Theorem for $\mathrm{Sm}\mathbb{Z}^{\#}$ – Polish groups

CONVERSE TO PRIKRY'S THM II

Group-free version

THM (Oz 2022)

Let
- Y be a σ-compact metric space
- X be a seperable locally cpt metric space
- $\phi : Y \times X \to X$ continuous

  $\forall y \in Y \ \cdots \ \phi(\{y\} \times P)$ nowhere dense whenever $P \subseteq X$ is nowhere dense

If $S \subseteq Y$ is $Smz^{\#}$ and $M \subseteq X$ is meager, then $\phi(S \times M)$ is meager

Coro

Let $\phi : G \curvearrowright X$ be an action of a σ-compact Polish group on a σ-compact space X.
If $S \subseteq G$ is $Smz^{\#}$ and $M \subseteq X$ is meager, then $\phi(S \times M)$ is meager.

Coro (Oz 2022)

Let G be a σ-compact Polish group.
Every $Smz^{\#}$ set in G is meager-additive.

GALVIN–MYCIELSKI–SOLOVAY THM – 2^ω

THM (Oz 2019) Every meager-additive set in 2^ω is $\mathrm{Sm}_{\mathbb{Z}}^{\#}$.

Proof
- $X \subseteq 2^\omega$ meager-additive, g gauge.
- We will prove: $\overline{\mathcal{H}}^g(X) = 0$.
- Recursion: $f \in \omega^{\uparrow\omega}$ .... $2^{f(k)}\, g(2^{-f(k+1)}) \leq 2^{-k}$

Shelah 1995: $\exists\, h \in \omega^\omega\ \exists\, y \in 2^\omega$

$\forall x \in X\ \forall^\infty n\ \exists k$ .... $h(n) \leq f(k) < h(n+1)$ &

$x \restriction [f(k), f(k+1)) = y \restriction [f(k), f(k+1))$

- $\mathcal{B}_k = \{ \langle p^\frown y \restriction [f(k), f(k+1)) \rangle : p \in 2^{f(k)} \}$
- $\mathcal{G}_n = \bigcup \{ \mathcal{B}_k : h(n) \leq f(k) < h(n+1) \}$

$\Longrightarrow$ $\forall x \in X\ \forall^\infty n\ \exists G \in \mathcal{G}_n\ x \in G$

$\longrightarrow \forall x \in X \ \forall^{\infty} n \ \exists G \in \mathcal{G}_n \quad x \in G$

$$\Longrightarrow \quad \mathcal{B} = \bigcup_{k \in \omega} \mathcal{B}_k = \bigcup_{n \in \omega} \mathcal{G}_n$$

is a γ-groupable cover

$\longrightarrow |\mathcal{B}_k| = 2^{f(k)}$

$\longrightarrow \operatorname{diam} B = 2^{-f(k+1)}$ for all $B \in \mathcal{B}_k$

$\longrightarrow 2^{f(k)} g\left(2^{-f(k+1)}\right) \leq 2^{-k}$

$$\longrightarrow \sum_{B \in \mathcal{B}} g(\operatorname{diam} B) = \sum_{k \in \omega} \sum_{B \in \mathcal{B}_k} g(\operatorname{diam} B)$$

$$= \sum_{k \in \omega} 2^{f(k)} g\left(2^{-f(k+1)}\right)$$

$$\leq \sum_{k \in \omega} 2^{-k} < \infty .$$

DEF Let $\mathcal{J}$ be an ideal on a Polish group $\mathbb{G}$.
A set $A \subseteq \mathbb{G}$ is <u>$\mathcal{J}$-additive</u> if

$$\forall J \in \mathcal{J} \quad AJ \in \mathcal{J}$$

→ $\mathcal{J}$ = meager sets ... meager-additive

→ $\mathcal{J} = \mathcal{E}$ (closed null sets on 2^ω) ...

... $\mathcal{E}$-additive

→ $\mathcal{J} = \mathcal{N}$ (null sets on 2^ω) ... null-additive

Nowik, Weiss 2002 : Problem 3

Is a set in 2^ω $\mathcal{E}$-additive iff it is (T') ?

THM (Dz 2019) A set $X \subseteq 2^\omega$ is $\mathcal{E}$-additive iff it is meager-additive.

$\longrightarrow$ Meager-additive $\Rightarrow$ $\mathcal{E}$-additive

Proof

- X is $\mathrm{Sm}_2^{\#}$
- If $E \in \mathcal{E}$ then $\pi^{-1}(X \times E) = 0$
- $\pi^{-1}(X + E) = 0$

  $(x, y) \mapsto x + y$ is Lipschitz
- $X + E \in \mathcal{E}$

$\mathcal{E}$-additive $\Rightarrow$ meager-additive

DEF $X \subseteq 2^\omega$ is <u>sharply $\mathcal{J}$-additive</u> if

$$\forall J \in \mathcal{J}\ \exists C \supseteq X\ F_\sigma \ \dots\ C + J \in \mathcal{J}$$

→ $\mathcal{E}$-additive $\Rightarrow$ sharply $\mathcal{E}$-additive

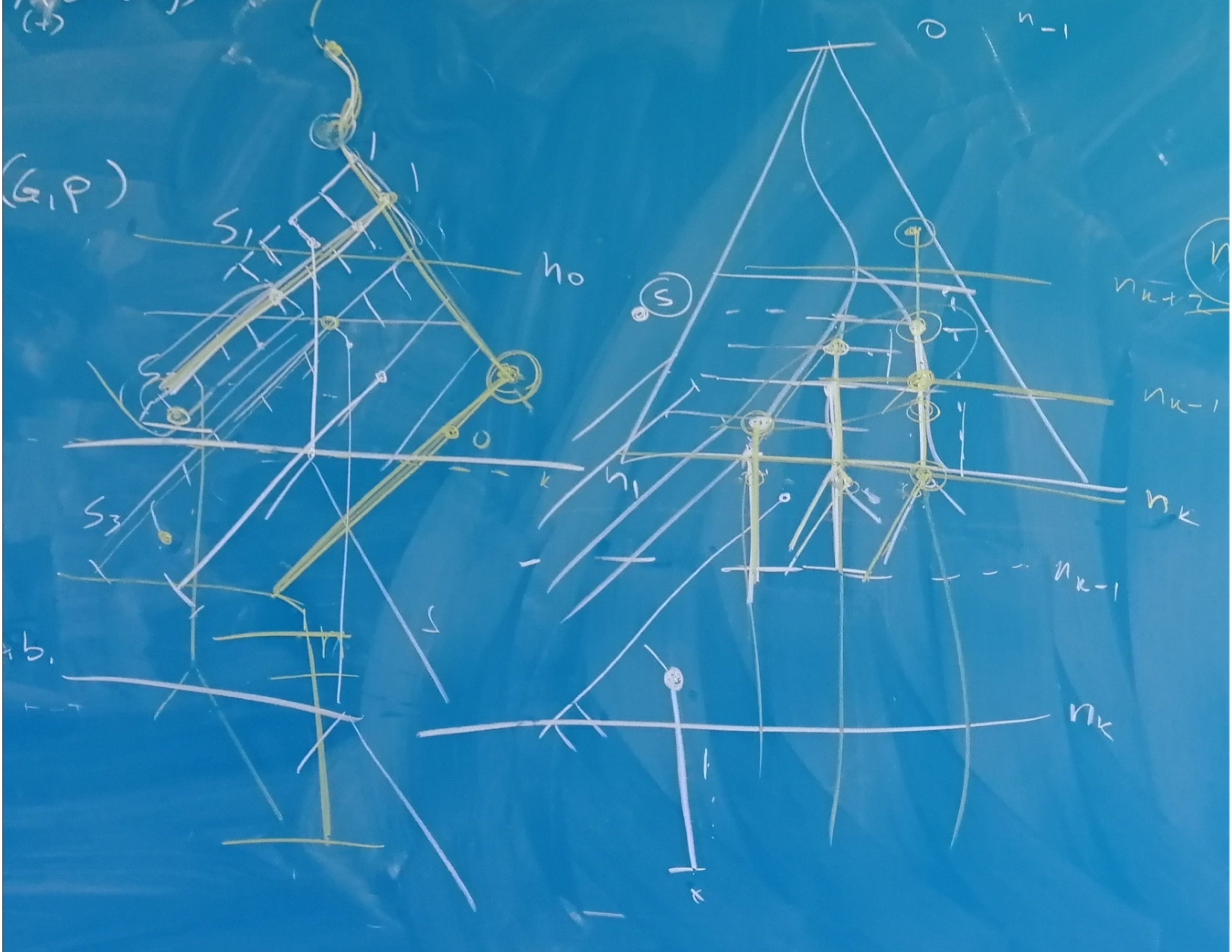
→ heavy combinatorics

→ sharply $\mathcal{E}$-additive $\Rightarrow$

$\Rightarrow$ sharply meager-additive

→ Pawlikowski's:

$$\forall M \in \mathcal{M}\ \exists E \in \mathcal{E}\ \forall X \subseteq 2^\omega\ X + E \in \mathcal{N} \Rightarrow X + M \neq 2^\omega$$


(+)
(G, P)
S_1
h_0
S_3
h_1
n_{k-1}
n_k
n_{k+2}
n_{k-1}
n_k

SUMMARY

THM (Oz 2019) Let $X \subseteq 2^\omega$. TFAE

- X is $Sm_2^\#$
- X is meager-additive
- X is sharply meager-additive
- X is $\mathcal{E}$-additive
- X is sharply $\mathcal{E}$-additive
- $\forall M \in \mathcal{M}\ \exists C \supseteq X\ F_\sigma \ldots\ C + M \neq 2^\omega$

Requires a lot of combinatorics specific for 2^ω.

MEAGER-ADDITIVE IN POLISH GROUPS

→ THE GOAL: GSM Theorem in Polish groups

→ Weiss 2005: Let $T : 2^\omega \to [0,1]$ be the canonical map. Then

$X \subseteq 2^\omega$ is meager-additive $\iff$

$\iff T(X) \subseteq [0,1]$ is meager-additive

→ COR: GSM for Smz# holds in $\mathbb{R}$.

THM (Oz 2022) Let G be a Polish group w/ left-invariant metric. Then

$X \subseteq G$ is sharply meager-additive $\Rightarrow$

$\Rightarrow X$ is Smz#

Proof: • X sharply meager-additive

• g gauge

• $\exists\, U \subseteq G$ G_δ dense ... $\mathcal{H}^g(U) = 0$

• $M = G \setminus U^{-1}$ ... $\exists\, C \supseteq X$ ... $CM \neq G$
σ-cpt

.... $\exists\, z$ $\quad X \subseteq C \subseteq zU$

• $\overline{\mathcal{H}}^g(X) \leq \overline{\mathcal{H}}^g(C) = \mathcal{H}^g(C) = \mathcal{H}^g(zU) = 0$

THM (Oz 2022) Let G be a σ-cpt Polish group.
TFAE for $X \subseteq G$

(i) X is $\mathrm{Sm}z^{\#}$

(ii) X is sharply meager-additive

GMS THEOREM FOR MEAGER-ADDITIVE

THM (Oz 2022) If G is a TSI Polish group, then $X \subseteq G$ is meager-additive $\Rightarrow$ X is $Sm_2^{\#}$

- $\rightarrow$ Grid in G that plays the role of $2^{<\omega}$
- $\rightarrow$ Base for meager sets based on the grid
- $\rightarrow$ Combinatorial rules for the base

COROLLARY Let G be a TSI σ-compact Polish group. TFAE for $X \subseteq G$:

(i) X is meager additive

(ii) X is sharply meager-additive

(iii) X is $Sm_2^{\#}$

PACKING MEASURE

DEF X is double-sharp measure zero ($Smz^{\times}$) if

$P^g(X) = 0$ for each gauge g

→ Borel-like definition ✓

→ Galvin-like game ✓

→ Product of $Smz^{\times}$ sets is $Smz^{\times}$

THM If $X \subseteq 2^{\omega}$ or $X \subseteq \mathbb{R}$:

X is $Smz^{\times} \iff X$ is null-additive

THM If $X \subseteq G$ (locally compact)

X is $Smz^{\times} \implies X$ is null-additive

GRAND UNIFIED THEORY

	measure	Borel	Game	Arrow
Sm_Z	$\mathcal{H}^g$	✓	✓	✓
$Sm_Z^{\#}$	$\bar{\mathcal{H}}^g$	✓	✓	✓
$Sm_Z^{\times}$	$\mathcal{P}^g$	✓	✓	×

GRAND UNIFIED THEORY

	measure	Borel	Game	Arrow
$Sm\mathbb{Z}$	$\mathcal{H}^g$	✓	✓	✓
$Sm\mathbb{Z}^{<\omega}$	$\pi\mathcal{H}^g$	✓	✓	✓
$Sm\mathbb{Z}^{\#}$	$\bar{\mathcal{H}}^g$	✓	✓	✓
(T')	ν^g	✓	✓	✓
$Sm\mathbb{Z}^{\times}$	ρ^g	✓	✓	×

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